



# A CRACKED BEAM FINITE ELEMENT FOR ROTATING SHAFT DYNAMICS AND STABILITY ANALYSIS

Saber El Arem, Habibou Maitournam

## ► To cite this version:

Saber El Arem, Habibou Maitournam. A CRACKED BEAM FINITE ELEMENT FOR ROTATING SHAFT DYNAMICS AND STABILITY ANALYSIS. *Journal of Mechanics of Materials and Structures*, 2008, 3 (5), pp.893-910. hal-00408873

**HAL Id: hal-00408873**

**<https://hal.science/hal-00408873>**

Submitted on 3 Aug 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A cracked beam finite element for rotating shaft dynamics and stability analysis

Saber El Arem    Habibou Maitournam

*Laboratoire de Mécanique des Solides CNRS (UMR 7649)  
Ecole Polytechnique, 91128 Palaiseau, France*

---

## Abstract

In this paper, a method for the construction of a cracked beam finite element is presented. The additional flexibility due to the cracks is identified from three-dimensional finite element calculations taking into account the unilateral contact conditions between the cracks lips as originally developed by Andrieux and Varé [2002]. Based on this flexibility which is distributed over the entire length of the element, a cracked beam finite element stiffness matrix is deduced. Considerable gain in computing efforts is reached compared to the nodal representation of the cracked section when dealing with the numerical integration of differential equations in structural dynamics. The stability analysis of a cracked shaft is carried using the Floquet theory.

*Key words:* breathing crack, beam, unilateral contact, finite element, rotor, Floquet, Stability.

---

## 1 Introduction

Because of the increasing need of energy, the plants installed by electricity supply utilities throughout the world are becoming larger and more highly stressed. Thus, the risk of turbogenerator shaft cracking is increasing also. Rotating shafts are omnipresent in Aeronautics, aerospace, automobile industries and in particular in the energy sector which is vital for any economic development.

Fatigue cracks is an important form of rotor damage which can lead to catastrophic failures if undetected early. They can have detrimental effects on the

---

*Email addresses:* `saber.el-arem@polytechnique.edu` (Saber El Arem ),  
`maitournam@lms.polytechnique.fr` ( Habibou Maitournam).

reliability of rotating shafts. According to [Bently and Muszynska \[1986\]](#), in the 70s and till the beginning of the 80s, at least 28 shaft failures due to cracks were registered in the USA energy industry. Thus, Since the 80s, the interest of researchers to characterize structures containing cracks is growing remarkably. Between the beginning of the 70s and the end of the 90s, more than 500 articles concerning the cracked structures was published [[Dimarogonas, 1996](#), [Bachschmid and Pennacchi, 2007](#)].

The problem of determination of the behavior of cracked structures has been tackled with for a long time. And the fact that a crack presence or a local defect in a structural member introduces a local flexibility that affects its vibration response was known long ago. This local flexibility is related to the strain energy concentration in the vicinity of the crack.

The study of cracked turbines rotors began in the industry of the energy in the USA with the works of [Dimarogonas \[1970, 1971\]](#). In Europe, the first works appeared only some years later in papers by [Gasch \[1976\]](#), [Mayes and Davies \[1976\]](#) and [Henry and Okah-Avae \[1976\]](#). These authors considered a simple model to account for the crack breathing mechanism to which we often make reference by "the switching crack model " or "the hinge crack model" : the crack is totally opened or totally closed.

The vibration analysis of cracked beams or shafts is a problem of great interest due to its practical importance. In fact, vibration measurements offer a non-destructive, inexpensive and fast means to detect and locate cracks. Thus, vibration behavior analysis and monitoring of cracked rotors has received considerable attention in the last three decades [[Gudmundson, 1982, 1983](#)]. It has, perhaps, the greatest potential since it can be carried without dismantling any part of the machine and usually online avoiding the costly downtime of turbomachinery.

[Zuo \[1992\]](#), and [Zuo and Curnier \[1994\]](#) also used a bilinear model to characterize the vibrational response of a cracked shaft with the aim of developing an online cracks detection method. They, first, examined the 1 degree of freedom (dof) system then studied the behavior of a system with 2 dof. By extending the Rosenberg normal mode notion for smooth and symmetric nonlinear systems to conwise linear systems, they defined the nonlinear modes of the bilinear system which were calculated numerically, and in certain simple cases, analytically. The application of this method to systems with high number of dof is complicated and would lead to high computation costs.

Bachschmid and his co-workers [[Bachschmid et al., 1980, 2002, 2004b](#)] examined the effects of the presence of a crack on the vibratory response of a rotor or a pump axis. Experimental and numerical models were proposed, the thermal effects on the crack breathing mechanism were taken into account

[Bachschnid et al., 2004b]. It was reported that the temperature distribution is not influenced by the presence of the crack unlike those of the stresses and deformations.

A comparison of various cracked beams models was presented by Friswell and Penny [2002]. The authors showed that in the low frequencies domain, simple models of the crack breathing mechanism and beam type elements are adequate to monitor structures health. However, the approach of modeling based on a bilinear dynamic system, which still makes reference, remains a simplistic approach which leads to some reserves about the quality of the quantitative results stemming from its exploitation.

A good review on the most relevant analytical, experimental and numerical works conducted in the three last decades and related to the cracked structures behavior were reported by Dimarogonas and Paipetis [1983], Entwistle and Stone [1990], Dimarogonas [1996], Wauer [1990], Gasch [1993], El Arem [2006].

Today, most of the works on the cracked shafts vibration analysis are concerned with investigating more deeply certain particular points such as the phase of acceleration or deceleration of the shaft, the passage through the critical speed or the coupling between diverse modes of vibrations to highlight parameters facilitating the online cracks detection when dealing with machines monitoring. These works such as those of Darpe et al. [2004], those of Jun et al. [1992] and those of Sinou and Lees [2005, 2007] where we find numerical and experimental results of their investigations on certain points such as aforementioned, remain faithful to the theoretical principles formalized in the 70s in some reference papers. In the other hand, remarkable and continuous progress of the computational tools allows the realization of successful three-dimensional modelings. So, it becomes possible to envisage an identification of the law of behavior of a cracked shaft section (breathing mechanism description) which frees from certain simplifying hypotheses and approximations made until now; this without degrading the costs of the calculations of the vibrational response.

The main objective of this work is the presentation of a method of construction of a cracked beam finite element. The stiffness variation of the element is deduced from three-dimensional finite element computations accounting for the unilateral contact between the cracks lips. Based on an energy approach, this method could be applied to cracks of any shape. The validation of the approach on a case of a cracked rotating shaft is then presented and its stability analysis is carried using the Floquet theory. The work of El Arem and Maitournam [2007] is taken back to show the general form of the stiffness matrix of the finite element and then facilitate the approximation of its terms.

## 2 Cracked rotors modeling: State of the art

The analysis of rotating machinery shafts behavior is a complex structural problem. It requires, for a relevant description, a fine and precise modeling of the rotor and cracks in order to allow the identification and calculation of the parameters characterizing their presence.

Researchers dealing with the problem of rotating cracked beam recognize its two main features, namely:

- the determination of the local flexibility of the beam cracked section;
- the consideration of the crack breathing mechanism responsible of the system nonlinearity: the system stiffness is depending on the cracked section position.

Most researchers agree with the application of the linear fracture mechanics theory to evaluate the local flexibility introduced by the crack [Gross and Srawley, 1965, Anifantis and Dimarogonas, 1983, Dimarogonas and Paipetis, 1983, Dimarogonas, 1996, Papadopoulos and Dimarogonas, 1987a,b,c, Papadopoulos, 2004]. Obviously, the first work was done in the early 1970 by Dimarogonas [1970, 1971] and Pafelias [1974] at the General Electric Company. There have been different attempts to quantify local effect introduced by cracks. The analysis of the local flexibility of a cracked region of structural element was quantified in the 1950s by Irwin [1957a,b], Bueckner [1958], Westmann and Yang [1967] by relating it to the Stress Intensity Factors (SIF). Afterwards, the efforts to calculate the SIF for different cracked structures with simple geometry and loading was duplicated [Tada et al., 1973, Bui, 1978].

For an elastic structure, the additional displacement  $\mathbf{u}$  due to the presence of a straight crack of depth  $a$  under the generalized loading  $\mathbf{P}$  is given by the Castigliano theorem

$$\mathbf{u} = \frac{\partial}{\partial \mathbf{P}} \int_0^a \mathbf{G}(a) da \quad (1)$$

$\mathbf{G}$  is the energy release rate defined in fracture mechanics and related to the SIF by the Irwin formula [Irwin, 1957a]. Then, the local flexibility matrix coefficients are obtained by

$$c_{ij} = \frac{\partial^2}{\partial P_i \partial P_j} \int_0^a \mathbf{G}(a) da, \quad 1 \leq i, j \leq 6 \quad (2)$$

Extra diagonal terms of this matrix are responsible for longitudinal and lateral vibrations coupling that can be with great interest when dealing with cracks detection.

In two technical notes of the NASA, Gross and Srawley [1964, 1965] computed

the local flexibility corresponding to tension and bending including their coupling terms. This coupling effect was observed by [Rice and Levy \[1972\]](#) in their study of cracked elastic plates for stress analysis.

Dimarogonas and his co-workers [[Dimarogonas, 1982](#), [Dimarogonas and Paipetis, 1983](#), [Dimarogonas, 1987, 1988](#)], and [Anifantis and Dimarogonas \[1983\]](#) introduced the full  $(6 \times 6)$  flexibility matrix of a cracked section. They noted the presence of extra diagonal terms which indicate the coupling between the longitudinal and lateral vibrations. [Papadopoulos and Dimarogonas \[1987a,b,c\]](#), and [Ostachowicz and Krawczuk \[1992\]](#) computed all the  $(6 \times 6)$  flexibility matrix of a Timoshenko beam cracked section for any loading case.

However, there are no results for the SIF for cracks on a cylindrical shaft. Thus, [Dimarogonas and Paipetis \[1983\]](#) have developed a procedure which is commonly used in FEM software: the shaft was considered as an assembly of elementary rectangular strips where approximation of the SIF using fracture mechanics results remains possible. The SIF are obtained by integration of the energy release rate on the crack tip. Although it offers the advantage of being easily applicable in a numerical algorithm, this method remains an approximation whose convergence remains to be checked. In fact, some numerical problems were noted [[Abraham et al., 1994](#), [Dimarogonas, 1994](#)] when the depth of the crack exceeds the section radius. Moreover, generalization of this method to any geometry of cracked section is complex, even impossible in the case of non connate multiple cracks affecting the same transverse section.

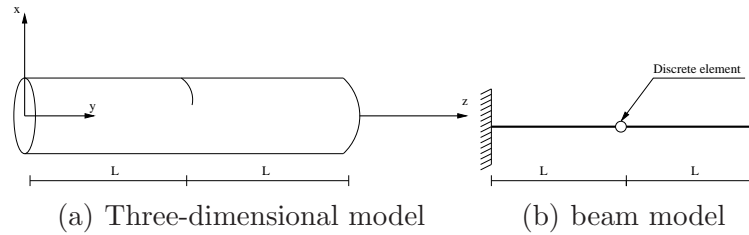


Fig. 1. *The cracked beam element modeling*

An original method for deriving a lumped cracked section beam model was proposed by [Varé and Andrieux \[2000\]](#) and [Andrieux and Varé \[2002\]](#). The procedure was designed by starting from three-dimensional computations and incorporating more realistic behavior on the cracks than the previous models, namely the unilateral contact conditions on the cracks lips and the breathing mechanism of the cracks under variable loading. The approach was validated experimentally by [Audebert and Voinis \[2000\]](#) and applied for the study of real cracked structures specially turbines.

The cracked element of Figure 1 is submitted to an end moment  $\mathbf{M}_{2L} = (M_x(2L), M_y(2L))$  at  $z = 2L$ . [Andrieux \[2000\]](#) has demonstrated certain properties of the problem elastic energy,  $W^*$ , leading to a considerable gain in

three-dimensional calculus needed for the identification of the behavior law. In particular, for a linear elastic material, under the small displacements and small deformations assumptions, and in the absence of friction on the cracks lips, the energy function could be written by distinguishing the contribution of the cracked section from that of the uncracked elements, in the form :

$$W^*(\mathbf{M}_{2L}) = W^*(\mathbf{M}) = W_s^*(\mathbf{M}) + w^*(\mathbf{M}) = \frac{L}{EI} \|\mathbf{M}\|^2 (1 + s(\Phi)) \quad (3)$$

where  $\mathbf{M} = (M_x, M_y)$  is the resulting couple of flexural moments at the cracked section,  $W_s^*(\mathbf{M})$  the total elastic energy of uncracked element submitted to the flexural moment  $\mathbf{M}$  and  $w^*(\mathbf{M})$  the additional elastic energy due to the presence of the cracked section. The loading direction angle is defined by  $\Phi = \text{atan}(\frac{M_y}{M_x})$ .  $E$  is the Young modulus and  $I$  quadratic moment of inertia. In this framework, the nonlinear behavior law of the discrete element modelling the cracked section is obtained by derivation of the function  $w^*(\mathbf{M})$  by  $\mathbf{M}$ . We obtain

$$[\boldsymbol{\theta}] = \begin{pmatrix} [\theta_x] \\ [\theta_y] \end{pmatrix} = \frac{2L}{EI} \begin{pmatrix} s(\Phi) & -\frac{1}{2}s'(\Phi) \\ \frac{1}{2}s'(\Phi) & s(\Phi) \end{pmatrix} \begin{pmatrix} M_x \\ M_y \end{pmatrix} \text{ with } s'(\Phi) = \frac{ds(\Phi)}{d\Phi} \quad (4)$$

However, for finite element computational codes in rotordynamics, the compliance function  $s(\Phi)$  is of low interest and a nonlinear relation of the form  $[\boldsymbol{\theta}] = f(\mathbf{M})$  is to be integrated in transient computations. Thus, [Andrieux and Varé \[2002\]](#) introduced some properties of the additional strain energy due to the cracked section,  $w([\boldsymbol{\theta}])$ . Thus,  $w$  could be written as a quadratic function of the rotations jumps as:

$$w([\theta_x], [\theta_y]) = \frac{EI}{4L} k(\varphi) \|[\boldsymbol{\theta}]\|^2 \text{ with } \varphi = \text{atan}\left(\frac{[\theta_y]}{[\theta_x]}\right) \quad (5)$$

By using the Légendre-Fenchel transform to establish the relation between the two energy functions  $w^*(\mathbf{M})$  and  $w([\boldsymbol{\theta}])$ , the stiffness function  $k(\varphi)$  is obtained from the compliance function  $s(\Phi)$  identified from three-dimensional calculus.

As described by [Andrieux and Varé \[2002\]](#), the behavior law is finally deduced from the derivation of  $w([\boldsymbol{\theta}])$  by  $[\boldsymbol{\theta}]$  as

$$\begin{pmatrix} M_x \\ M_y \end{pmatrix} = \frac{EI}{2L} \begin{pmatrix} k(\varphi) & -\frac{1}{2}k'(\varphi) \\ \frac{1}{2}k'(\varphi) & k(\varphi) \end{pmatrix} \begin{pmatrix} [\theta_x] \\ [\theta_y] \end{pmatrix} \text{ with } k'(\varphi) = \frac{dk(\varphi)}{d\varphi} \quad (6)$$

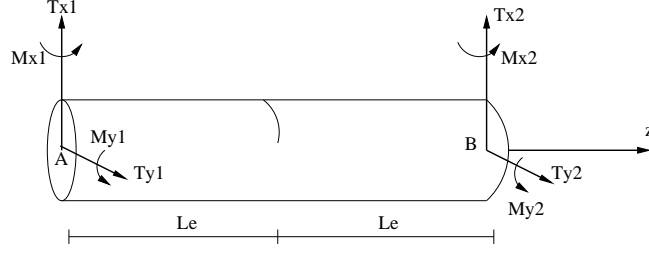


Fig. 2. *The cracked beam finite element.*

### 3 A cracked beam finite element construction

There are two procedures to introduce the local flexibility generated by a cracked section when dealing with the numerical integration of differential equations in dynamics. The first technique considers the construction of a stiffness matrix exclusively for the cracked section by computing the inverse of the flexibility matrix. However, for small cracks, the additional flexibility is very small and, consequently, the corresponding stiffness coefficient are extremely large leading to high numerical integration costs and convergence problems [El Arem, 2006].

The second procedure, adopted here, consists in the construction of a cracked finite element stiffness matrix, which is later assembled with the other uncracked elements of the system. Thus, the elastic energy due to the cracks is distributed over the entire length of the cracked element. This method has been used in works by Bachschmid et al. [2004a] and Saavedra and Cuitino [2001].

The studies of Verrier and El Arem [2003], El Arem et al. [2003], Varé and Andrieux [2005] and El Arem [2006] showed that the shear effects on the breathing mechanism of the cracks is insignificant and will be neglected in this study.

Let consider the cracked finite element of length  $2L_e$ , circular transverse section of diameter  $D$  and quadratic moment of inertia  $I$ , cf. Figure 2.

First, we clamp all the displacements of node  $A$  and establish a relation of the form

$$\mathbf{u} = \mathcal{S}(\mathbf{f}) \cdot \mathbf{f} \quad (7)$$

$\mathbf{f} = \{T_{x2}, T_{y2}, M_{x2}, M_{y2}\}^t$  and  $\mathbf{u} = \{u_{x2}, u_{y2}, \theta_{x2}, \theta_{y2}\}^t$  denote, respectively, the loading and displacements vectors at the end section ( $z = 2L_e$ ).  $\mathcal{S}(\mathbf{f})$  represents the compliance matrix of the structure.

At the cracked section ( $z = L_e$ ), the internal efforts are given by :



$$\begin{cases} T_x = T_{x_2} \\ T_y = T_{y_2} \\ M_x = M_{x_2} - L_e T_y \\ M_y = M_{y_2} + L_e T_x \end{cases} \quad (8)$$

The breathing mechanism of the cracks is governed by the flexural moment direction  $\Phi = \text{atan}(\frac{M_y}{M_x})$  at the cracked section. The elastic energy of the cracked element could be written as

$$W^*(\mathbf{f}) = W_s^*(\mathbf{f}) + w^*(\mathbf{M}) = W_s^*(\mathbf{f}) + \frac{L}{EI} \|\mathbf{M}\|^2 s(\Phi) \quad (9)$$

$W_s^*(\mathbf{f})$  denotes the elastic energy of the uncracked finite element of the same geometry and submitted to the same loading conditions. By using equation (3), the additional elastic energy due to the cracked section is given by

$$w^*(\mathbf{M}) = \frac{L}{EI} \|\mathbf{M}\|^2 s(\Phi) \quad (10)$$

where  $L$  is the half length of the three-dimensional element used to identify the compliance function  $s(\Phi)$  as described below.

The nonlinear relation between the applied efforts and the resulting displacements at the end section ( $z = 2L_e$ ) are obtained by derivation of the function  $W^*$  by  $\mathbf{f}$ . Thus, by using (8), we write

$$\mathbf{u} = \mathcal{S}(\mathbf{f}) \cdot \mathbf{f} = \mathcal{S}(\Phi) \cdot \mathbf{f} \quad (11)$$

where

$$\mathcal{S}(\Phi) = \mathcal{S}_0 + \frac{2L}{EI} \begin{pmatrix} L_e^2 s(\Phi) & -\frac{L_e^2}{2} s'(\Phi) & \frac{L_e}{2} s'(\Phi) & L_e s(\Phi) \\ \frac{L_e^2}{2} s'(\Phi) & L_e^2 s(\Phi) & -L_e s(\Phi) & \frac{L_e}{2} s'(\Phi) \\ -\frac{L_e}{2} s'(\Phi) & -L_e s(\Phi) & s(\Phi) & -\frac{1}{2} s'(\Phi) \\ L_e s(\Phi) & -\frac{L_e}{2} s'(\Phi) & \frac{1}{2} s'(\Phi) & s(\Phi) \end{pmatrix} \quad (12)$$

$\mathcal{S}_0$  denotes the compliance matrix of an uncracked beam element of length  $2L_e$ .

Let consider  $\{u_{B/A}\}$  the relative displacement of node  $B$  in rapport of node  $A$ . It verifies the relation

$$\{T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t = (\mathcal{S}(\Phi))^{-1} \{u_{B/A}\} \quad (13)$$

From equilibrium conditions of the element of Figure 2, the internal forces in  $B$  can be expressed in terms of those in  $A$  as:

$$\begin{cases} T_{x_1} = -T_{x_2} \\ T_{y_1} = -T_{y_2} \\ M_{x_1} = -M_{x_2} + 2L_e T_{y_2} \\ M_{y_1} = -M_{y_2} - 2L_e T_{x_2} \end{cases} \quad (14)$$

or, written in a matrix form:

$$\{T_{x_1}, T_{y_1}, M_{x_1}, M_{y_1}, T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t = \mathbf{\Pi}_1 \{T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t \quad (15)$$

$$\text{with} \quad \mathbf{\Pi}_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2L_e & -1 & 0 \\ -2L_e & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (16)$$

In addition, when writing  $\{u_{B/A}\}$  in the form

$$\{u_{B/A}\} = \{u_{B/A}^1, u_{B/A}^2, u_{B/A}^3, u_{B/A}^4\}$$

we obtain

$$\begin{cases} u_{x_2} = u_{B/A}^1 + u_{x_1} + 2L_e \theta_{y_1} \\ u_{y_2} = u_{B/A}^2 + u_{y_1} - 2L_e \theta_{x_1} \\ \theta_{x_2} = u_{B/A}^3 + \theta_{x_1} \\ \theta_{y_2} = u_{B/A}^4 + \theta_{y_1} \end{cases} \quad (17)$$

or, in a matrix form

$$\{u_{B/A}\} = \mathbf{\Pi}_2 \{u_{x_1}, u_{y_1}, \theta_{x_1}, \theta_{y_1}, u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t \quad (18)$$

$$\text{where} \quad \mathbf{\Pi}_2 = \begin{pmatrix} -1 & 0 & 0 & -2L_e & 1 & 0 & 0 & 0 \\ 0 & -1 & 2L_e & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (19)$$

Comparing with equation (16), it can be seen that:

$$\mathbf{\Pi}_2 = \mathbf{\Pi}_1^t$$

Moreover, the cracked beam finite element stiffness matrix  $\mathbf{K}_{ef}$  verifies

$$\begin{aligned} \{T_{x_1}, T_{y_1}, M_{x_1}, M_{y_1}, T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t = \\ \mathbf{K}_{ef} \{u_{x_1}, u_{y_1}, \theta_{x_1}, \theta_{y_1}, u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t \end{aligned} \quad (20)$$

Substituting equation (15) into equation (20) gives

$$\mathbf{\Pi}_1 \{T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t = \mathbf{K}_{ef} \{u_{x_1}, u_{y_1}, \theta_{x_1}, \theta_{y_1}, u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t \quad (21)$$

Then, substituting equation (13) into equation (21), results in

$$\mathbf{\Pi}_1 (S(\Phi))^{-1} \{u_{B/A}\} = \mathbf{K}_{ef} \{u_{x_1}, u_{y_1}, \theta_{x_1}, \theta_{y_1}, u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t \quad (22)$$

Finally, using equation (18) leads to:

$$\mathbf{K}_{ef} = \mathbf{\Pi}_1 (\mathcal{S}(\Phi))^{-1} \mathbf{\Pi}_1^t \quad (23)$$

In this relation, the stiffness matrix appears as depending on the loading efforts represented by angle  $\Phi$ . However, in a finite element code, it is preferable to express relation (23) as a function of the problem's unknowns, that is the nodal displacements. We start by writing :

$$(\mathcal{S}(\Phi))^{-1} = \mathcal{K}(\varphi_e) = \mathcal{K}_0 - \mathbf{K}_e(\varphi_e) \quad (24)$$

to distinguish the stiffness matrix of an uncracked element of length  $2L_e$  and inertial moment  $I$ ,  $\mathbf{\Pi}_1 \mathcal{K}_0 \mathbf{\Pi}_1^t$ , from the matrix modelling the cracked section presence  $\mathbf{\Pi}_1 \mathbf{K}_e(\varphi_e) \mathbf{\Pi}_1^t$ .  $\varphi_e$  is the angle given by  $\varphi_e = \text{atan}(\frac{\theta_{y_2} - \theta_{y_1}}{\theta_{x_2} - \theta_{x_1}})$  and  $\mathcal{K}_0$  by Lalanne and Ferraris [1990]

$$\mathcal{K}_0 = \mathcal{S}_0^{-1} = \frac{EI}{2L_e(1+a)} \begin{pmatrix} \frac{3}{L_e^2} & 0 & 0 & -\frac{3}{L_e} \\ 0 & \frac{3}{L_e^2} & \frac{3}{L_e} & 0 \\ 0 & \frac{3}{L_e} & 4+a & 0 \\ -\frac{3}{L_e} & 0 & 0 & 4+a \end{pmatrix} \quad (25)$$

$a = \frac{12EI}{4\mu kSL_e^2}$  is the shearing effects coefficient. For an Euler-Bernoulli beam element,  $a$  is zero.

Equation (24) leads to

$$\mathbf{K}_e(\varphi_e) = \mathcal{K}_0 - (\mathcal{S}(\Phi))^{-1} = \frac{EI}{2L} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{xx}(\varphi_e) & k_{xy}(\varphi_e) \\ 0 & 0 & k_{yx}(\varphi_e) & k_{yy}(\varphi_e) \end{pmatrix} \quad (26)$$

where

$$\begin{cases} k_{xx}(\varphi_e) = k_{yy}(\varphi_e) = \frac{L^2(4L_e s(\Phi) + 4L s^2(\Phi) + L s'^2(\Phi))}{L_e(4L_e^2 + 8LL_e s(\Phi) + 4L^2 s^2(\Phi) + L^2 s'^2(\Phi))} \\ k_{xy}(\varphi_e) = -k_{yx}(\varphi_e) = -\frac{2L^2 s'(\Phi)}{4L_e^2 + 8LL_e s(\Phi) + 4L^2 s^2(\Phi) + L^2 s'^2(\Phi)} \end{cases} \quad (27)$$

When  $L = L_e$ , we obtain

$$\begin{cases} k_{xx}(\varphi_e) = k_{yy}(\varphi_e) = \frac{4s(\Phi) + 4s^2(\Phi) + s'^2(\Phi)}{4 + 8s(\Phi) + 4s^2(\Phi) + s'^2(\Phi)} \\ k_{xy}(\varphi_e) = -k_{yx}(\varphi_e) = -\frac{2s'(\Phi)}{4 + 8s(\Phi) + 4s^2(\Phi) + s'^2(\Phi)} \end{cases} \quad (28)$$

Using the three-dimensional calculus conducted on the cracked element of Figure 1, we identify the compliance function  $s(\Phi)$  as described previously. Then, we determine the stiffness matrix  $\mathbf{K}_e(\varphi_e)$  terms by using relation (26).

We have noticed that:

$$k_{xy}(\varphi_e) = -\frac{1}{2}k'_{xx}(\varphi_e) \quad (29)$$

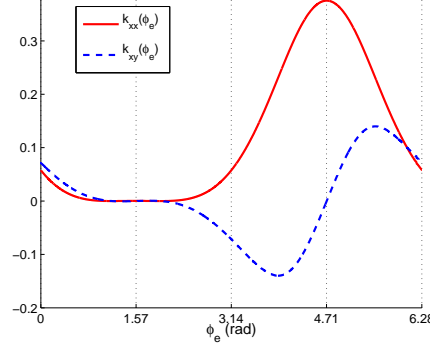


Fig. 3.  $\mathbf{K}_e(\varphi_e)$  terms for a straight crack of depth  $a = \frac{D}{2}$ ,  $L_e = L = 2D$ .

Thus,  $\mathbf{K}_e(\varphi_e)$  can be written in the form:

$$\mathbf{K}_e(\varphi_e) = \frac{EI}{2L} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{xx}(\varphi_e) & -\frac{1}{2}k'_{xx}(\varphi_e) \\ 0 & 0 & \frac{1}{2}k'_{xx}(\varphi_e) & k_{xx}(\varphi_e) \end{pmatrix} \quad (30)$$

For a straight crack of depth  $a = \frac{D}{2}$ , these terms are shown by Figure 3. We notice small variations on  $[0, \frac{\pi}{2}]$  interval, however, on  $[\frac{\pi}{2}, 2\pi]$ , these variations becomes important but remain regular. The crack is totally closed when  $\varphi_e = \frac{\pi}{2}$ , image of  $\Phi = \frac{\pi}{2}$ . It opens totally at  $\varphi_e = \frac{3\pi}{2}$ , image of  $\Phi = \frac{3\pi}{2}$ .  $\Phi$  is the angle defined in section 2 by  $\Phi = \text{atan}(\frac{M_y}{M_x})$

#### 4 Validation of the approach

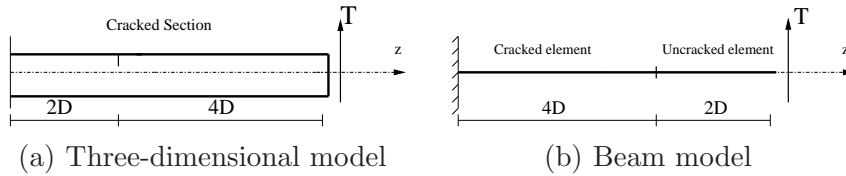


Fig. 4. *Three-dimensional and beam modeling of the system.*

In order to validate the stiffness beam finite element matrix construction method presented above, we propose to compare the three-dimensional modeling calculus results to those obtained using a beam modeling of the cracked structure of Figure 4. The cylinder element of axis  $(oz)$ , of diameter  $D = 0.5m$ , and total length  $L_t = 3m$ , is clamped at its end  $z = 0$  and submitted at the other to couple of efforts  $\mathbf{T} = (T_x = \cos(\alpha), T_y = \sin(\alpha))$  with  $\alpha$  varying from 0 to  $2\pi$ . The cracked section is located at  $z = 1m$ . The crack is straight with

depth  $a = 0.5D$ , cf. Figure 4(a). The three-dimensional finite element calculus take into account the unilateral contact conditions between the crack lips. The beam model consists of two beam finite elements: the first one is a cracked beam finite element of length  $2m$ . The second is a classic beam finite element of length  $1m$ , cf. Figure 4(b). Figure 5 shows excellent results concordance.

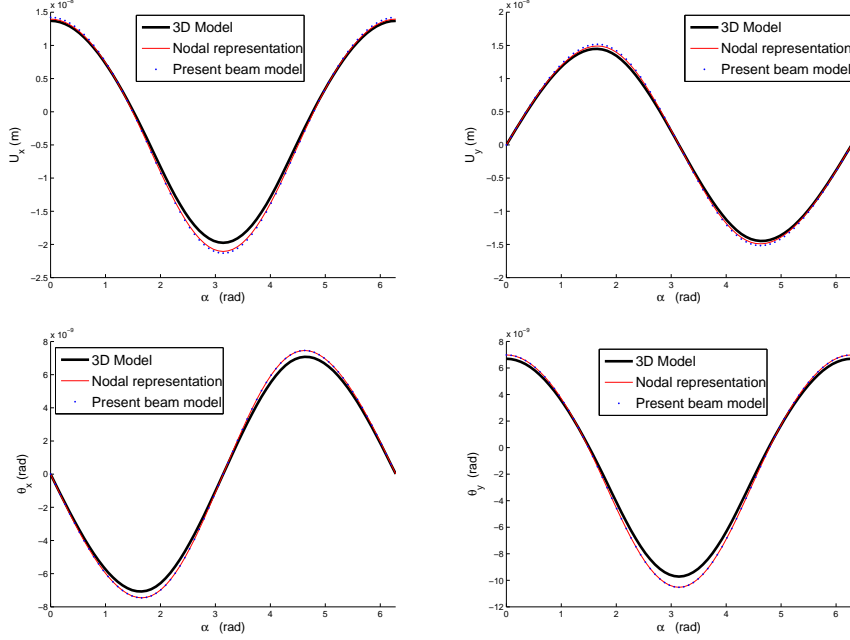


Fig. 5. *Model comparison to three-dimensional results and nodal modelling,  $\frac{a}{D} = 0.50$ .*

## 5 Vibratory response of a cracked shaft

This section is deserved to explore the vibratory response of a shaft with a cracked section under own weight effects. The system is composed of a beam of distributed mass  $m$ , circular section  $S$  and diameter  $D = 0.20m$ . The cracked section is located at mid-span. The structure is simply supported at node 1 and node 6 and submitted to the effects of its own weight. The shaft is rotating at the frequency  $\Omega$  about  $(oz)$  axis. The system is divided into five elements of length  $l = 4D$ , cf. Figure 6. The elements 1 – 2, 2 – 3, 4 – 5 and 5 – 6 represent the structure uncracked parts. 3 – 4 is a cracked beam finite element.

At  $t = 0$ , the crack is totally open, cf. Figure 6(b). We suppose that the stiffness matrix remains constant between two instants  $t_n = nh$  and  $t_{n+1} = (n+1)h$  where  $h$  is the time step used for the numerical integration of the dynamical system. For the validity of this approximation, the time steps should be relatively small compared to the excitation period  $(\frac{1}{\Omega})$ . In the low frequency

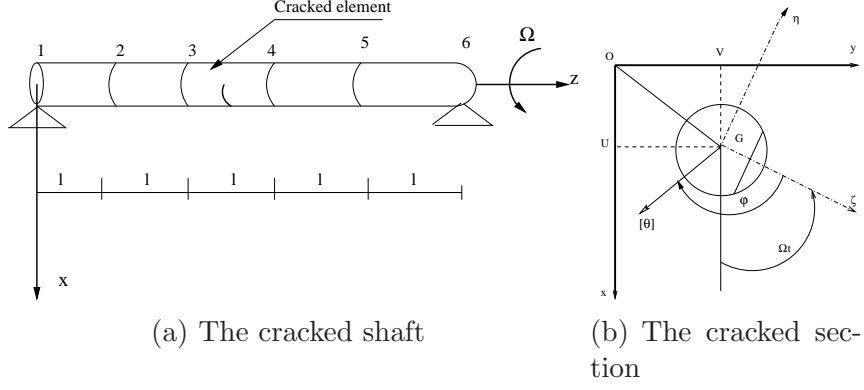


Fig. 6. *Finite element modeling of the cracked structure.*

domain, this difficulty is easily overcome. Thus, the stiffness matrix is updated at the end of each integration step. The HHT method is adopted for the numerical integration of the obtained system [Hilber et al., 1977] with:

$$\alpha = \frac{1}{3}, \gamma = \frac{1}{2} + \alpha \text{ et } \beta = \frac{1}{4}(1 + \alpha)^2$$

which corresponds, in the linear analysis, to an unconditionally stable scheme with maximum precision. With this modeling, the unilateral contact conditions between the crack lips are taken into account exactly. In fact, when the crack is totally closed, we obtain

$$\mathbf{K}_e(\varphi_e) = \mathbf{0}$$

and the cracked element stiffness matrix is the one of an uncracked element. Overmore, the time step used here is  $10^{-3}s$  which is at least 20 to 100 times smaller than the ones used for penalization-implicit approach when the cracked section is modeled by a nodal element (element of length zero). In this later approach, the time step depends on the value given to the penalization constant and is always less than  $10^{-5}s$  to ensure calculus convergence.

The vibratory response shows the superharmonic resonance phenomena presence when the rotating frequency passes through entire divisions of the critical speed  $w_1$ . Thus, for

$$\xi \approx \frac{w_1}{n} \text{ with } n \in \mathbb{N}$$

the shaft orbit and the phase diagram are composed of  $n$  interlaced loops (cf. Figure 7 where the viscous damping  $d$  is of 5%). Also, the vibratory amplitude of harmonic  $n \times$  reaches, at this rotating frequency, higher levels, cf. Figure 8.

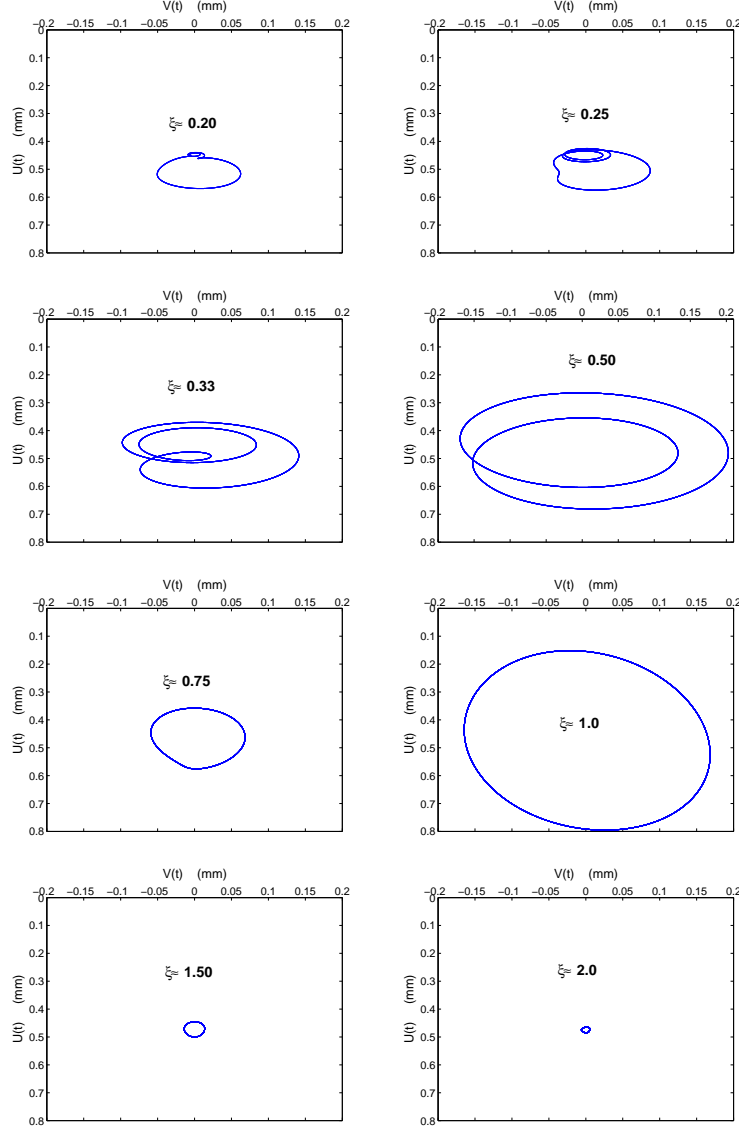


Fig. 7. Examples of node 2 orbits,  $\frac{a}{D} = 0.50$ ,  $d=0.05$ .

## 6 Stability analysis

The stability of a cracked shaft is analyzed using the Floquet method Floquet [1879], Nayfeh and Mook [1979], Nayfeh and Balachandran [1994]. This method was used by Gasch [1976], Meng and Gasch [2000] and El Arem and Nguyen [2006] for the stability analysis of a two parameters cracked rotating shaft. The first step of this section consists of approximating the  $\mathbf{K}_e(\varphi_e)$  terms by a classical function of the crack depth  $a$  and angle  $\varphi_e$ . We have noticed that the function  $k_{xx}(\varphi_e)$  for different straight tip crack depths shows that it



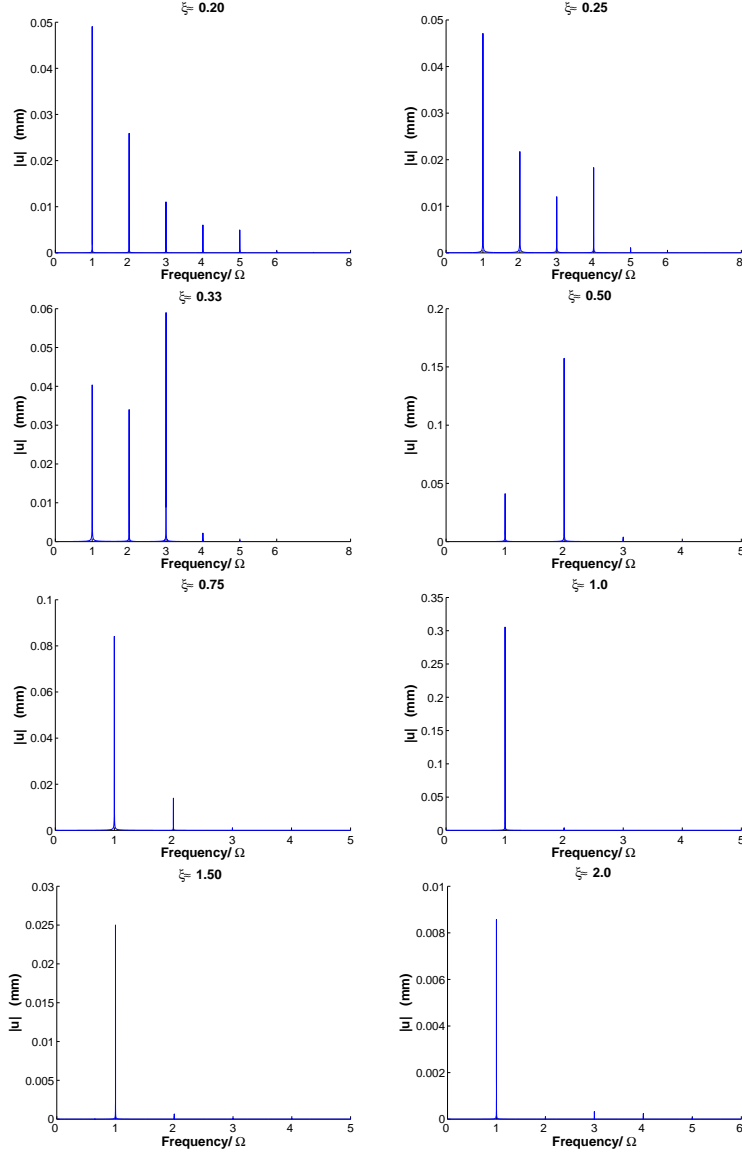


Fig. 8. Examples of node 3  $u(t)$  amplitude spectra,  $\frac{a}{D} = 0.50$ ,  $d=0.05$ .

could be approximated by:

$$k_{xx}(\varphi_e) \approx k_{max} \sin^4\left(\frac{\varphi_e}{2} - \frac{\pi}{4}\right) e^{(\sin(\frac{\varphi_e}{2} - \frac{\pi}{4}))^4} \quad (31)$$

where  $k_{max}$  is given by

$$k_{max} \approx 3.43 \left(\frac{a}{D}\right)^{2.73} \quad (32)$$

These approximations remain satisfactory for cracks of depth going up to 30 percent of the diameter of the shaft, cf. Figure 9.

By using the Floquet theory, we have investigated, numerically, the stability of a cracked rotating shaft with a straight tip crack at mid-span, cf. Figure

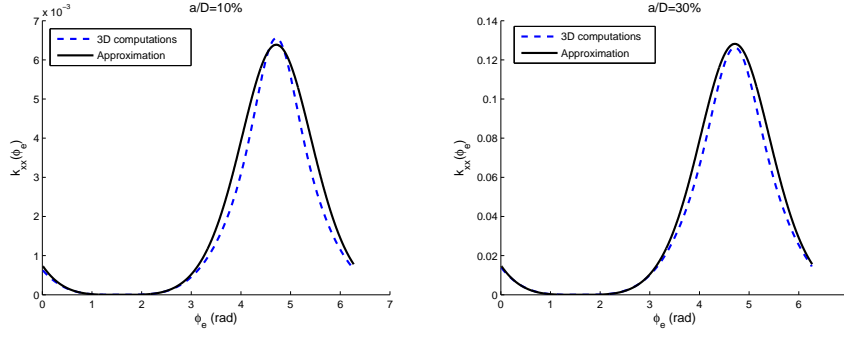


Fig. 9.  $k_{xx}(\varphi_e)$  approximation for different straight crack depths.

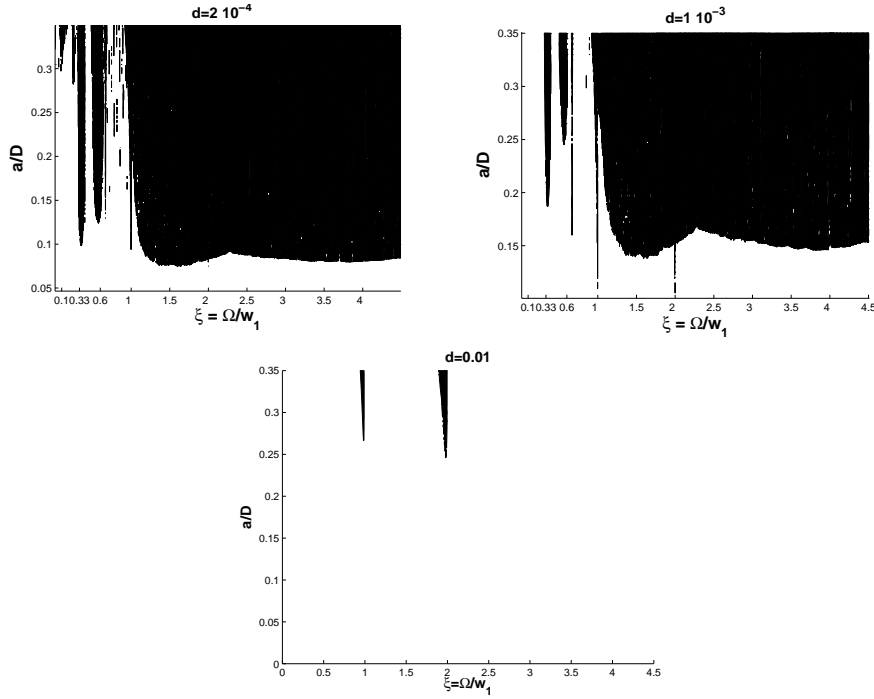


Fig. 10. Stable and instable (hatched) regions.

6. Results, cf. Figure 10, show three principal instability areas: the first zone corresponds to  $(0 < \xi < 0.5)$  superharmonic resonance phenomenon. The second is located around the exact resonance  $(\xi = 1)$  and the third area (around  $\xi = 2$ ) corresponds to subharmonic resonance. It's important to note that even for weak viscous damping ( $d \approx 1\%$ ) the stability of the cracked shaft is only slightly affected. The zones of instabilities appear for  $\Omega$  near the first critical speed ( $\xi \approx 1$ ) and twice the critical speed ( $\xi \approx 2$ ) and correspond to deep cracks ( $\frac{a}{D} > 25\%$ ).

## 7 Conclusions

A method for the construction of a cracked beam finite element is presented. The crack breathing phenomenon is finely described since the flexibility due to the presence of the cracks is deduced from three-dimensional finite element calculations taking into account of the conditions of unilateral contact between the cracks lips as originally developed by [Andrieux and Varé \[2002\]](#). The precise descriptions of the loss of stiffness and of the progressive closure or opening of the cracks are of fundamental importance.

The approach is quite simple and comprehensive and can be applied to any geometry of cracks. It is important to note the considerable gain in computational efforts when comparing with the use of the technique of penalization.

Indeed this technique, used when the cracked section is modeled by a nodal element [[El Arem, 2006](#)], leads to the appearance of very high numerical frequencies (without physical signification). The time steps considered for the temporal numerical integration of the dynamic system are then very small and the computation costs are, consequently, very important. Compared to the nodal representation of the cracked section and the technique of penalization to consider the conditions of contact between the cracks lips, this model has the following advantages:

- taking into account in an exact way of the phenomenon of cracks breathing mechanism,
- reduction of the costs of calculations from 20 to 100 times.

In the study of the cracked shafts, the researchers often supposed, to reduce the difficulty of the problem, that the amplitude of the vibrations due to the presence of cracks is weak compared to those of the vibrations due to permanent loads ([El Arem and Nguyen \[2006\]](#), [Gasch \[1976\]](#), [Mayes and Davies \[1976\]](#), [Mayes and Davies \[1980\]](#), [Davies and Mayes \[1984\]](#), [Henry and Okah-Avae \[1976\]](#), [Gasch \[1993\]](#)). The present model is more general and allows to overcome limitations due to such an assumption.

The stability analysis of the cracked shaft of Figure 6 was carried using the Floquet theory. Figure 10 shows stable and instable zones in the  $(\xi, \frac{a}{D})$  plan for different viscous damping coefficients. We have noticed that the instability zones disappear for  $d \geq 3\%$ . It can be deduced that, for real turbines shafts, where the viscous damping is about 3% to 4% [[Lalanne and Ferraris, 1990](#)], the effects of a cracked section at mid-span on the stability of the structure is negligible when  $\frac{a}{D}$  is less than 35% .

**Acknowledgements** The authors would like to thank Huy Duong Bui for useful discussions.

## References

- O. N. L. Abraham, J. A. Brandon, and A. M. Cohen. Remark on the determination of compliance coefficients at the crack section of a uniform beam with circular cross section. *J. Sound Vib.*, 169(2):570–574, 1994.
- S. Andrieux. Détermination de la loi de comportement en flexion bi-axée d’une poutre fissurée avec prise en compte du contact sur les fissures (in french). rapport interne HP-50/99/018/A, EDF-DER, 2000.
- S. Andrieux and C. Varé. A 3d cracked beam model with unilateral contact-application to rotors. *European Journal of Mechanics, A/Solids*, 21:793–810, 2002.
- N. Anifantis and A. D. Dimarogonas. Identification of peripheral cracks in cylindrical shells. In *ASME-Wint. Ann. Meet.*, Boston, Mass., USA, 1983.
- S. Audebert and P. Voinis. Comportement dynamique de rotors avec fissuration transverse: modélisation et validation expérimentale (in french). In *13me colloque Vib. Chocs et bruits*, Lyon, France, 2000.
- N. Bachschmid and P. Pennacchi. Crack effects in rotordynamics. *Mechanical Systems and Signal*, doi:10.1016/j.ymssp.2007.11.003, 2007.
- N. Bachschmid, N. Diana, and B. Pizzigoni. The influence of unbalance on cracked rotors. In *Conf. Vib. Rotat. Mach.*, U.K, 1980.
- N. Bachschmid, P. Pennacchi, and A. Vania. Identification of multiple faults in rotor systems. *Journal of Sound and Vibration*, 254(02):327–366, 2002.
- N. Bachschmid, Pennacchi P., Tanzi E., and Vania A. Cracks in rotating shafts: experimental behaviour, modelling and identification. In *SURVEIL-LANCE 5 fifth international conference on acoustical and vibratory surveillance methods and diagnostic techniques*, Senlis, France, 2004a.
- N. Bachschmid, P. Pennacchi, E. Tanzi, and S. Audebert. Transverse crack modeling and validation in rotor systems including thermal effects. *Int. J. of Rotat. Mach.*, 10(4):253–263, 2004b.
- D.E. Bently and A. Muszynska. Detection of rotor cracks. In *15th Turbomachinery Symposium*, Texas, 1986.
- H. F. Bueckner. The propagation of cracks and the energy of elastic deformation. *Trans. ASME*, 80:1225–1229, 1958.
- Huy Duong Bui. *Mécanique de la rupture fragile (in french)*. Masson, 1978.
- A. K. Darpe, K. Gupta, and A. Chawla. Transient response and breathing behaviour of a cracked jeffcott rotor. *J. Sound Vib.*, 272:207–243, 2004.
- W.G.R. Davies and I.W. Mayes. The vibrational behaviour of multi-shaft, multi-bearing system in the presence of the propagating transverse crack. *J. Vib. Acous. Stress Reliab. Design*, 106:146–153, 1984.
- A. D. Dimarogonas. A fuzzy logic neural network structured expert system

- shell for diagnosis and prognosis- users manual. In *EXPERTS*, Clayton laboratories, St Louis, Missouri, USA, 1987.
- A. D. Dimarogonas. Author's reply. *J. Sound and Vibration*, 169:575–576, 1994.
- A. D. Dimarogonas. Vibration of cracked structures: A state of the art review. *Engineering Fracture Mechanics*, 55(5):831–857, 1996.
- A. D. Dimarogonas. Dynamic response of cracked rotors. Internal report, General Electric Co., Schenectady NY, 1970.
- A. D. Dimarogonas. Dynamics of cracked shafts. Internal report, General Electric Co., Schenectady NY, 1971.
- A. D. Dimarogonas. Crack identification of in aircraft structures. In *First National aircraft conf.*, Athens, 1982.
- A. D. Dimarogonas. A general purpose rotor dynamic analysis program- users manual. In *RODYNA*, Clayton laboratories, St Louis, Missouri, USA, 1988.
- A. D. Dimarogonas and S. A. Paipetis. *Analytical methods in rotor dynamics*. Applied science Publishers, 1983.
- S. El Arem. *Vibrations non-lineaires des structures fissurées: application aux rotors de turbines (in french)*. PhD thesis, Ecole Nationale des Ponts et Chaussées, 2006.
- S. El Arem and H. Maitournam. Un élément fini de poutre fissurée: application la dynamique des arbres tournants (in french). *European Journal of Computational Mechanics*, 16(5):643–663, 2007.
- S. El Arem and Q. S. Nguyen. A simple model for the dynamical behavior of a cracked rotor. In *Advances in Geomaterials and Structures*, pages 393–398, Hammamet, 2006.
- S. El Arem, S. Andrieux, C. Varé, and P. Verrier. Loi de comportement en flexion d'une section de poutre fissurée avec prise en compte des effets de cisaillement (in french). In *6ème colloque nationale en calcul des structures*, volume II, pages 223–230, Giens, 2003.
- R. D. Entwistle and B. J. Stone. Survey of the use of vibration methods in the assessment of component geometry. *Vib. and Noise Meas. Pred. and Cont., Inst. engineers, Australia*, 90:210–217, 1990.
- G. Floquet. Sur les équations différentielles linéaires coefficients périodiques. *Ann. Scien. Ecole Nor. Sup.*, 2me série:3–132, 1879.
- M.I. Friswell and J.E.T. Penny. Crack modelling for structural health monitoring. *International Journal of Structural Health Monitoring*, 1:139–148, 2002.
- R. Gasch. Dynamical behavior of a simple rotor with a cross-sectional crack. In *Vibrations in rotating machinery*, pages 123–128, I. Mech. E. Conference, London, 1976.
- R. Gasch. A survey of the dynamic behavior of a simple rotating shaft with a transverse crack. *J. Sound and vibration*, 160:313–332, 1993.
- B. Gross and J. E. Srawley. Stress intensity factors for a single-edge-notch tension specimen by boundary collocation of a stress function. Tech. Note D-2395, NASA, 1964.

- B. Gross and J. E. Srawley. Stress intensity factors for a single-edge-notch specimen in bending or combined bending and tension by boundary collocation of a stress function. Tech. Note D-2603, NASA, 1965.
- P. Gudmundson. Eigenfrequency changes of structures due to cracks, notches or other geometrical changes. *J. Mech. Phys. Solids*, 30(5):339–353, 1982.
- P. Gudmundson. The dynamical behavior of slender structures with cross-sectional cracks. *J. Mech. Phys. Solids*, 31:329–345, 1983.
- T. A. Henry and Okah-Avae. Vibrations in cracked shafts. In *Vibrations in rotating machinery*, pages 15–19, Inst. Mech. E. Conference, London, 1976.
- H. M. Hilber, T.J.R. Hughes, and R.L. Taylor. Improved numerical dissipation for time integration algorithms in structural dynamics. *Earthquake Engineering and Structural Dynamics*, 5:283–293, 1977.
- G. R. Irwin. Analysis of stresses and strains near the end of a crack traversing a plate. *J. App. Mech.*, 24:361–364, 1957a.
- G. R. Irwin. Relation of stresses near a crack to the crack extension force. In *9th Cong. App. Mech.*, Brussels, 1957b.
- O. S. Jun, H. J. Eun, Y.Y. Earmme, and C.W. Lee. Modelling and vibration analysis of a simple rotor with a breathing crack. *J. Sound and Vibration*, 155(2):273–290, 1992.
- M. Lalanne and G. Ferraris. *Rotordynamics Prediction in Engineering*. John Wiley & Sons, England, 1990.
- I.W. Mayes and W.G.R. Davies. The vibrational behaviour of a rotating shaft system containing a transverse crack. In *Vibrations in rotating machinery*, pages 53–65, Inst. Mech. E. Conference, London, 1976.
- I.W. Mayes and W.G.R. Davies. A method of calculating the vibrational behaviour of coupled rotating shafts containing a transverse crack. In *Vibrations in rotating machinery*, pages 17–27, London, 1980. Inst. Mech. E. Conference.
- G. Meng and R. Gasch. Stability and stability degree of a cracked flexible rotor supported on journal bearings. *J. Vibration and Acoustics, Transactions of ASME*, 122:116–125, 2000.
- A. H. Nayfeh and B. Balachandran. *Applied nonlinear dynamics*. Wiley series in nonlinear science, 1994.
- A. H. Nayfeh and D. T. Mook. *Nonlinear oscillations*. John Wiley & Sons, 1979.
- W. M. Ostachowicz and M. Krawczuk. Coupled torsional and bending vibrations of a rotor with an open crack. *Arch. App. Mech.*, 62:191–201, 1992.
- T. Pafelias. Dynamic response of a cracked rotor. Technical Information Series DF-74-LS-79, General Electric Co., 1974.
- C. A. Papadopoulos. Coupling of bending and torsional vibration of a cracked timoshenko beam. *J. Sound and Vibration*, 278:1205–1211, 2004.
- C. A. Papadopoulos and A. D. Dimarogonas. Coupled longitudinal and bending vibrations of a rotating shaft with an open crack. *J. Sound vibration*, 117:81–93, 1987a.
- C. A. Papadopoulos and A. D. Dimarogonas. Coupling of bending and tor-

- sional vibration of a cracked timoshenko beam. *Ing. Arch.*, 57:496–505, 1987b.
- C. A. Papadopoulos and A. D. Dimarogonas. Stability of cracked rotors in the coupled vibration mode. In *ASME-11th Bien. Conf. Mech. Vib. Noise*, pages 25–34, Boston, Mass., USA, 1987c.
- J. R. Rice and N. Levy. The part-through surface crack in an elastic plate. *J. App. Mech.*, 39:185–194, 1972.
- P. N. Saavedra and L. A. Cuitino. Crack detection and vibrational behavior of cracked beams. *Comput. Struct.*, 79:1451–1459, 2001.
- J. J. Sinou and A. W. Lees. The influence of cracks in rotating shafts. *J. Sound Vib.*, 285:1015–1037, 2005.
- J. J. Sinou and A. W. Lees. A non-linear study of a cracked rotor. *European Journal of Mechanics- A/Solids*, 26:152–170, 2007.
- H. Tada, P. Paris, and G. Irwin. *The stress analysis of cracks handbook*. Del Research Corporation, Hellertown, pennsylvania, USA, 1973.
- C. Varé and S. Andrieux. Modélisation d’une section de poutre fissurée-application aux rotors de turbine. *Revue Française de Mécanique*, 2000(2): 91–97, 2000.
- C. Varé and S. Andrieux. Modélisation d’une section de poutre fissurée en flexion: prise en compte des efforts tranchants (in french). In *7ème colloque nationale en calcul des structures*, volume II, pages 29–34, Giens, 2005.
- P. Verrier and S. El Arem. Calcul d’un modle de rotor fissuré avec prise en compte de l’effort tranchant-cas de la flexion plane (in french). rapport interne HT-65/03/009/A, EDF-DER, 2003.
- J. Wauer. On the dynamics of cracked rotors: A literature survey. *Applied Mechanical Reviews*, 43(1):13–17, 1990.
- R. A. Westmann and W. H. Yang. Stress analysis of cracked rectangular beams. *J. APP. Mech.*, 32:639–701, 1967.
- L. Zuo. *Etude du comportement dynamique des systmes linéaires par morceaux- contribution la détection des fissures dans les arbres de machines tournantes*. PhD thesis, EPFL, 1992.
- L. Zuo and A. Curnier. Nonlinear real and complex modes of conewise linear systems. *J. Sound and Vib.*, 174(3):289–313, 1994.